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Quenching problem of globally coupled bistable stochastic systems with finite size

Y. Zhang[1,a] and G. Hu[2,1]

[1] Physics Department, Beijing Normal University, Beijing 100875, China
[2] Center of Theoretical Physics, CCAST (World Laboratory), Beijing 8730, China

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Abstract. The transient process of globally coupled bistable systems from an unstable state to metastable state (*i.e*, quenching process) is studied analytically for small noise intensity. The influences of noise intensity and system size on the system evolution are investigated. The problem of a large number of coupled Langevin equations is reduced to a simple problem of a one-dimensional ordinary differential equation, subject to a white noise with intensity explicitly given. The analytical results are fully confirmed by direct numerical computations.

PACS. 05.45.+j Fluctuation phenomena, random processes, and Brownian motion

In the last several decades, the effect of noise on nonlinear systems has attracted constant interest in physics, chemistry, biology and almost all fields of natural science. Rather recently, the investigation of stochastic extended systems has become one of the central focuses on the noise problem. In this respect, the following model of globally coupled bistable systems [1–3]

$$\dot{x_i}\ (t) = ax_i(t) - x_i^3(t) + \mu X(t) + \Gamma_i(t)$$
$$i = 1, 2, ..., N$$
$$X(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t), \quad \langle \Gamma_i(t)\rangle = 0,$$
$$\langle \Gamma_i(t)\Gamma_j(t')\rangle = 2D\delta_{ij}\delta(t-t') \qquad (1)$$

is one of the models most extensively studied, where $X(t)$ is the spatial average of the realizations of x_i at time t. In this letter, we fix $a = 1$, this stochastic spatially extended system with global coupling is an important prototype for describing many practical systems, such as neural networks, multi-mode solid lasers, coupled Josephson junctions, and other biological and physiological systems [4–6]. In spite of a variety of works having dealt with equations (1), up to date, two problems of fundamental significance for applications have still escaped consideration. First, most works have considered only the asymptotic state of equations (1), the transient process of (1) from an unstable state to stable or metastable state, that has been extensively investigated as the quenching problem for the single site bistable system $\dot{x} = x - x^3 + \Gamma(t)$ [7–9], has never been considered either numerically or analytically. For the coupled extended systems, the quenching problem is related to important phenomena of clusterization and order formation and so on, and should be clarified. Second, all works published so far, which analytically treated equations (1), have identically taken the large system size limit $N \to \infty$, the finite size effect has never been analytically investigated. Actually, the second problem is closely related to the first one since, as it will be shown, in the infinite system size limit the time needed for the system to evolve from an unstable state to a metastable state is infinity, then no quenching process can be observed for infinite size.

In this letter we will analytically study the quenching problem of equations (1) for large but finite system size. In the present stage any explicit analytical results can be available only in the limit $D \to 0$. For convenience, in our analysis we consider also $0 < \mu \ll 1$ and $N \gg 1$, then our analysis is based on the following conditions

$$1 \gg \mu \gg D,\ N \gg 1. \qquad (2)$$

The inequality $\mu \gg D$ guarantees bistability of system (1) for the macroscopic variable X [3,5]. It will be shown that the analytical results obtained can be well confirmed by numerically running the original spatiotemporal stochastic systems (1) at small but finite D and μ, and large but finite N.

Before computing equations (1) we first specify the quenching problem in our case. For $t < 0$ we set $\mu = 0$, then all the N stochastic bistable systems evolve independently, then we obtain $\langle x_i(t=0)\rangle = 0$, $\langle \Delta x_i^2(t=0)\rangle = 1$, $[\Delta x_i(t) = x_i(t) - \langle x_i(t)\rangle]$. According to the Large Number

[a] e-mail: xdwen@email.bnu.edu.cn

Theory, we get

$$\langle X(t=0)\rangle = 0,$$
$$\langle \Delta X^2(t=0)\rangle = \frac{1}{N}\sum_{i=1}^{N}\langle \Delta x_i^2(t=0)\rangle = \frac{1}{N}$$
$$\rho(X,0) \approx \sqrt{\frac{N}{2\pi}}\exp(-\frac{NX^2}{2}) \quad (3)$$

where $\rho(X,0)$ is the probability distribution of X at time $t = 0$, and $\langle X\rangle$and $\langle \Delta X^2\rangle$ are the ensemble average of the macroscopic variable X and its variance, respectively. After $t \geq 0$ we switch on the global coupling among the bistable sites (*e.g.*, by feeding back the total output beam for the laser array, or by including a resistance in the electrical circuit of the Josephson junction array). In the condition (2), the state $X = 0$ is now unstable, the system will eventually evolve to one of the two stable states $X \approx \pm 1$ [5]. The quenching problem is to investigate the evolution from $X = 0$ to $X = \pm 1$.

It is well known that under the condition (2), the continuous bistable systems (1) can be reduced to two-state ones, and then the coupled stochastic bistable systems can be simplified to the following coupled master equations [1]

$$\dot{P}_i^{\pm} = -R^{\pm}P_i^{\pm} + R^{\mp}P_i^{\mp}, \quad P_i^{+} + P_i^{-} = 1,$$
$$i = 1, 2, ..., N \quad (4)$$

where P_i^+ and P_i^- are the probabilities for the ith site to take the state $+1$ and -1, respectively, and R^+ (R^-) is the transition rate from $+1$ (-1) state to -1 ($+1$) state

$$R^{\pm} = \frac{1}{\sqrt{2}\pi}\exp\left[-\frac{1}{4D} \mp \frac{\mu X(t)}{D}\right],$$
$$R = R^+ + R^- = r_0\mathrm{ch}(\frac{\mu X(t)}{D}),$$
$$r_0 = \frac{\sqrt{2}}{\pi}\exp(-\frac{1}{4D})\,. \quad (5)$$

By averaging equations (4), we have $\langle X(t)\rangle = \frac{1}{N}\sum\limits_{i=1}^{N} x_i(t) = \frac{1}{N}\sum\limits_{i=1}^{N}(P_i^+ - P_i^-)$, and we can further reduce the N coupled master equations to a single ordinary differential equation

$$\langle \dot{X}\ (t)\rangle = -r_0\mathrm{ch}\left[\frac{\mu X(t)}{D}\right]\langle X(t)\rangle + r_0\mathrm{sh}\left[\frac{\mu X(t)}{D}\right]\cdot \quad (6)$$

The derivation from (4) to (6) is exact. In all previous publications analytically treating (1) and (4), authors always used the identity $X(t) = \langle X(t)\rangle$, which is valid only for $N \to \infty$. It is worthwhile remarking that the evolution from the unstable state $X = 0$ to the stable state $X = 1$ (or $X = -1$) can never happen if one sets $X = \langle X\rangle$, since the time for the system to stay at the unstable point can be infinitely long.

In order to describe the practical spontaneous ordering process we should consider the finite system size effect by setting

$$X(t) = \langle X(t)\rangle + \eta(t),\ \langle \eta(t)\rangle = 0. \quad (7)$$

The statistical property of $\eta(t)$ can be computed, based on the assumption: around the stable and unstable fixed points of $\langle X(t)\rangle$, the variation of the macroscopic variable $X(t)$ is much slower than the variation of the microscopic variable x_i, and then we can consider $X(t)$ to be constant when we compute the variance of x_i (so-called adiabatic approximation treatment). A direct computation gives

$$\langle \eta(t)\eta(t')\rangle = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\langle \Delta x_i(t)\Delta x_j(t')\rangle$$
$$= \frac{1}{N^2}\langle \Delta x_i(t)\Delta x_i(t')\rangle$$
$$= \frac{1}{N[\mathrm{ch}\frac{\mu X(t)}{D}]^2}\exp(-R\mid t-t'\mid) \quad (8)$$

where we use $\langle \Delta x_i(t)\Delta x_j(t')\rangle = 0$ for $i \neq j$, and the computation of $\langle \Delta x_i^2(t)\rangle$ is based on equations (4). On the other hand, far from the fixed points of $\langle X(t)\rangle$, the stochastic force caused by $\eta(t)$ is negligibly smaller than the macroscopic force in (6), then for the entire evolution process from the disordered state $X(t) \approx 0$ to the ordered states $X(t) \approx \pm 1$, we can safely and consistently apply equation (8). By inserting (7) to (6) we arrive at a single stochastic equation

$$\dot{X}\ (t)a = -r_0\mathrm{ch}\left(\frac{\mu X(t)}{D}\right)X(t) + r_0\mathrm{sh}\left(\frac{\mu X(t)}{D}\right)$$
$$+ R\eta(t) + \frac{d\eta(t)}{dt}$$
$$\rho(X,0) = \sqrt{\frac{N}{2\pi}}\exp\left(-\frac{NX^2}{2}\right) \quad (9)$$

where $\eta(t)$ is an effective colored noise [10–13] having zero mean and exponentially decay correlation given in (8). It seems that the derivative of noise $\frac{d\eta(t)}{dt}$ leads to some difficulty for analytical treatment. Fortunately, this term does not cause any problem due to the cancellation in equation (9). By identifying

$$\frac{d\eta(t)}{dt} = -R\eta + R\Delta(t) \quad (10)$$

we can reduce equation (9) to

$$\dot{X(t)} = -r_0\mathrm{ch}\left(\frac{\mu X(t)}{D}\right)X(t) + r_0\mathrm{sh}\left(\frac{\mu X(t)}{D}\right)$$
$$+ r_0\mathrm{ch}\left(\frac{\mu X(t)}{D}\right)\Delta(t)$$
$$\langle \Delta(t)\Delta(t')\rangle = 2D'\delta(t-t')$$
$$D' = \frac{1}{N\,r_0\left[\mathrm{ch}(\frac{\mu X(t)}{D})\right]^3}\,. \quad (11)$$

Now the N stochastic ($\infty > N \gg 1$) coupled Brownian motions (1) are reduced to a much simpler single ordinary differential equation. It is rather interesting to see that the simple one-dimensional colored noise problem of

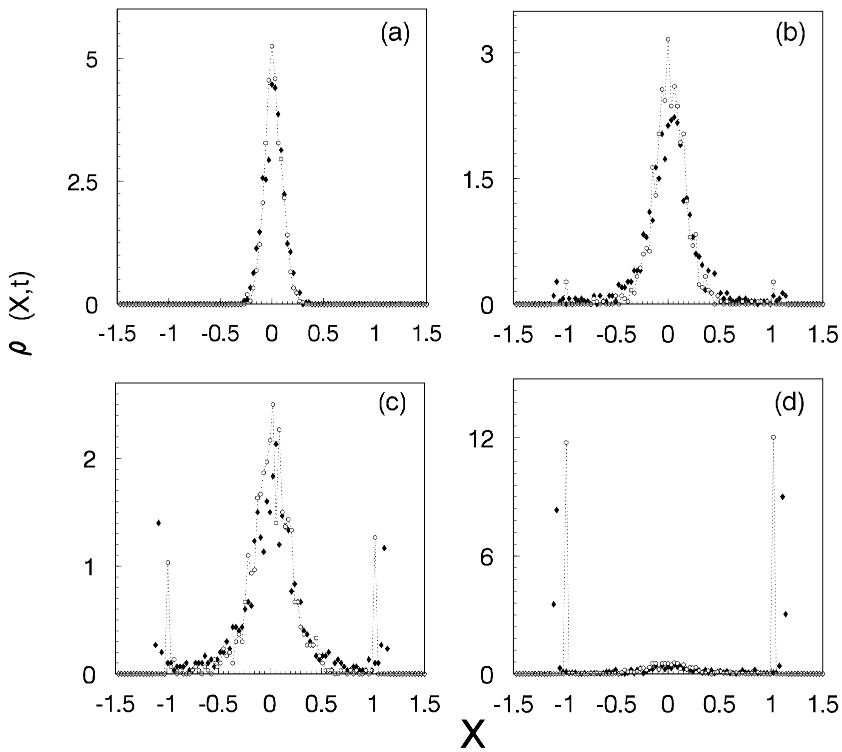

Fig. 1. Probability distributions $\rho(X,t)$ at different time t's. (a) $t = 0$, (b) $t = 600$, (c) $t = 900$, (d) $t = 2700$. (the same μ is taken for all the following figures) $\mu = 0.25$ for $t > 0$ and $\mu = 0$ for $t < 0$. The computation starts at $t = -5000$. $D = 0.03$, $N = 100$. Diamonds and circles represent the results of equations (1, 11), respectively. 1000 runs are carried out for presenting the data. The agreement of equations (11, 1) is strikingly good.

equation (9) is now further reduced to an even simpler multiplicative white noise problem equation (11). This reduction is very general in the two-state approximation, irregarding the particular form of the system. The statistical features of the noise are known explicitly. The computation time of (11) is incomparably smaller than that of equations (1). However, as we will see afterwards, the results of (11) agree with those of (1) quantitatively for small D and μ and large N. The qualitative agreement of (1) and (11) can be observed even for relatively large D and μ. Therefore, a thorough investigation of the reduced equation (11) can shed light on deep understanding of the spontaneous ordering of the very complicated set of equations (1). For instance, from equations (8) and (11) we find that the roles played by finite system size N, noise intensity D and coupling μ are the following:

1. The influences of finite N on the quenching process come from two aspects: first, reducing the system size N can increase the fluctuation around the unstable point $X = 0$ at $t = 0$ (see $\rho(X,0)$ in Eq. (9)) that definitely speeds the transition from disorder states (the unstable state $X = 0$ corresponds to disordered occupations of the subsystems at $x_i = \pm 1$) to ordered states (x_i take identically $+1$, or -1); second, reducing N can surely increase the intensity of effective noise $\eta(t)$ (and also the intensity of the white noise $\Delta(t)$) that also makes the evolution away from the unstable state faster.
2. It is interesting to see that changing the noise intensity of $\Gamma_i(t)$ in equations (1) does not greatly affect either the initial probability preparation of $\rho(X,0)$, or the variance of the effective noise $\langle \eta(t)^2 \rangle$, but it does sensitively influence the correlation time of the effective noise $1/R$, and the deterministic relaxation time of equation (11) $\tau_R = \frac{1}{[(\frac{\mu}{D}-1)R]}$. These two characteristic times are important for the quenching problem. In particular, the relaxation time τ_R is of crucial importance for determining the rate for the system to leave the macroscopic unstable point $X = 0$.
3. Increasing the coupling μ definitely reduces the relaxation time τ_R.

In order to confirm our theory, and verify all the reductions from equations (1) to (11), we numerically compute equations (1, 11) and compare these two types of solutions by varying N and D. In all the following numerical calculations we fix $\mu = 0.25$ which is not very small at all. In Figures 1 we take $D = 0.03$ and $N = 100$, and show the probability distributions $\rho(X,t)$ for different times. Diamonds and circles indicate the numerical results of equations (1, 11), respectively. For presenting the probability distributions, averages of 1000 runs are used. It is interesting to point out that the characteristic changes of the probability distributions from one-peak profiles to two-peak ones for both equations happen at approximately the same time. This shows that equations (1, 11) provide identical quenching times and the agreements between the two types of solutions are confirmed. Larger N and smaller D correspond to larger relaxation time. Some small mismatches in Figure 1b-1d are due to relatively large μ; that shifts the stable state of equations (1) to $\pm\sqrt{1+\mu}$ which deviate from the approximate states of equation (11), ± 1.

In Figures 2, 3 we plot the quenching time T *vs.* D (N) by fixing $N = 100$ ($D = 0.06$). T is defined as the average

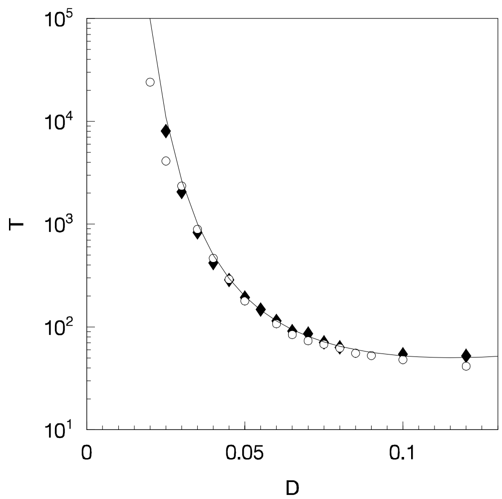

Fig. 2. The quenching time T plotted *vs.* D at $N = 100$. T is defined as the mean first passage time for $X(t)$ to first cross the boundary $X = \pm 0.8$. Diamonds and circles are computed by running equations (1, 11), respectively. Average over 200 runnings is taken for obtaining each datum. The solid line shows the optimal matching continual curve.

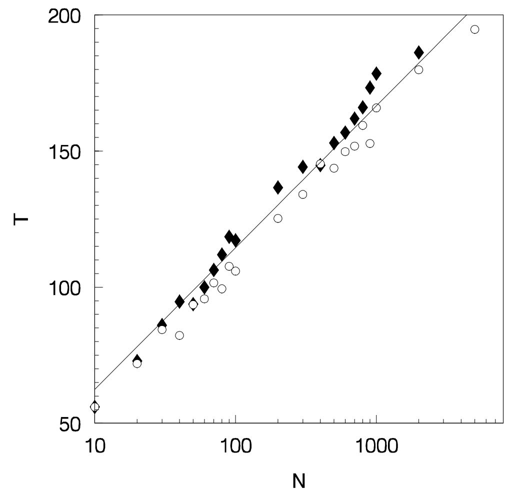

Fig. 3. The same as in Figure 2 but $D = 0.06$, T plotted *vs.* N. T increases in a manner of logarithm of N. The solid line has the same meaning as in Figure 2.

first passage time for $X(t)$ to pass $X = \pm 0.8$, computed by running equations (1) (diamonds) and (11) (circles), respectively. The averages are taken over 200 runs. Both results coincide with each other satisfactorily. Nevertheless, for some very large N and very small D, the computation of equations (1) is so time consuming that there is no chance for us to directly run equations (1), but we can safely use the circles to perfectly represent the behavior of equations (1) because the above reductions from (1) to (11) work better for smaller D and larger N.

In conclusion we have analyzed the quenching process of a globally coupled array of stochastic bistable systems. The spontaneous evolution from an original disordered state to an ordered state is investigated both analytically and numerically. A reduction of a set of N ($N \gg 1$) coupled Langevin equations to a single-variable Langevin equation has been carried out, and the intensity and the correlation time of the effective Langevin force is explicitly given. On one hand, this reduction makes analytical treatment possible, on the other hand it enormously saves numerical computing time (to several orders), that makes numerical investigations of very large N and very small D possible.

In this letter we defined the quenching problem by the sudden change of the coupling μ from $\mu = 0$ to $\mu > 0$ in equations (1). We can use another also very practical definition of quenching: sudden change of a in equations (1) from $a < 0$ to $a = 1$ with coupling $\mu > 0$ unchanged. In this case the system evolution has two stages: first, various sites make quenching from $x_i \approx 0$ to $x_i \approx \pm 1$ with $X = \frac{1}{N}\sum_{i=1}^{N} x_i \approx 0$ unchanged; second X evolves from $X \approx 0$ to $X \approx \pm 1$ through the probability transitions of various sites. The second stage is exactly what we analyzed throughout the letter. Since the time needed for the first stage is incomparably shorter than that for the second stage, the difference between these two quenching definitions (by changing μ, or by changing a) is not important. Therefore, the approach used in this letter is suitable for the general quenching problems of globally coupled stochastic systems.

Since the problem of large number of coupled Brownian motions and the related problems of phase separations, order formations and clusterizations are of great significance for practical physical, chemical systems and interacting biology systems, we hope the study in this letter may stimulate further investigations of time-dependent evolutions of coupled stochastic systems.

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